C3 Q203 Chapter 16B: LESSON 1 – Surface Integrals

The surface S:

Parametrization of the surface S: $x = x(u,v) \quad y = y(u,v) \quad z = z(u,v)$

Vector function of a surface S: $\mathbf{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$

Area of a surface S: $A = \iint_S dS = \iint_{uv} |\mathbf{r}_u \times \mathbf{r}_v| dA$

Mass of a surface S: $mass = \iint_S f(x,y,z) dS = \iint_{uv} f(x(u,v), y(u,v), z(u,v)) |\mathbf{r}_u \times \mathbf{r}_v| dA$

(with density $f(x,y,z)$)

Flux of **F** through a surface S: $\Phi = \iint_S \mathbf{F} \cdot \hat{n} dS = \iint_{uv} \mathbf{F}(x(u,v), y(u,v), z(u,v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v) dA$

I. What are two ways to get a normal vector to the surface $z = f(x, y)$ at point P?

1.

2.

II. With Respect To … (w.r.t)

A. Area $\longrightarrow$ $A = \int_a^b f(x)dx$

$f(x) =$ sum-able quantity contributing to area
$dx =$ infinite increment of change

B. Arc Length $\longrightarrow$ $s = \int_C ds = \int_C \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$

$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$ = sum-able quantity contributing to length
$dt =$ infinite increment of change

C. Surface Area $\longrightarrow$

III. Find the parametrization $\mathbf{r}(u,v)$ (of the surface S)

A. The surface is a function:

Example:

General:

B. The surface is a circular cylinder:

Example:

General:

C. The surface is a sphere:

Example:

General:

IV. Find $(\mathbf{r}_u \times \mathbf{r}_v)$ and $|\mathbf{r}_u \times \mathbf{r}_v|$

A. The surface is a function:

B. The surface is a circular cylinder (therefore the radius a is fixed):

C. The surface is a sphere (therefore the radius a is fixed):

16.6#21. Find the parametric representation for the part of the hyperboloid $x^2 + y^2 - z^2 = 1$ that lies to the right of the xz-plane.

16.6 #24. Find the parametric representation for the part of the sphere $x^2 + y^2 + z^2 = 16$ that lies between the planes z = -2 and z = 2.

16.6 #38 Find the area of the part of the plane $2x + 5y + z = 10$ that lies inside the cylinder $x^2 + y^2 = 9$.

16.6 #42 Find the area of the part of the surface $z = 1 + 3x + 2y^2$ that lies above the triangle with vertices (0, 0), (0, 1), (2, 1).

16.6 #44. Find the area of the part of the paraboloid $x = y^2 + z^2$ that lies inside the cylinder $y^2 + z^2 = 9$

16.7-1: Evaluate $\iint_S y\,dS$ where S is the surface $z = x + y^2$, $0 \le x \le 1$, $0 \le y \le 2$

16.7-2: Evaluate $\iint_S x^2 dS$ where S is the unit sphere $x^2 + y^2 + z^2 = 4$.

16.7-1: Evaluate $\iint_S z\,dS$ where S is the surface whose sides S1 are given by $x^2 + y^2 = 1$, whose bottom S2 is the disk $x^2 + y^2 \le 1$ in the plane z = 0, and whose top S3 is the part of the plane $z = 1 + x$ that lies above S2.

C3: Q203 LESSON 2 – FLUX and $\iint_S \mathbf{F}\cdot d\mathbf{S}$

Flux of **F** through a positive oriented surface S:

$$\Phi = \iint_S \mathbf{F}\cdot \hat{n}\,dS = \iint_{uv} \mathbf{F}\big(x(u,v), y(u,v), z(u,v)\big)\cdot\left(\mathbf{r}_u \times \mathbf{r}_v\right)dA$$

Concept of Flux:

If a surface can be described by $z = f(x, y)$ then the surface is considered positively oriented if the z-component of the normal vectors is positive. If a surfaces can be describes by $x = f(y, z)$ or $y = f(x, z)$, then the surface is considered positively oriented if the *x* component and *y* component, respectively, of the respective normal vectors are positive. If a surface is closed, then a positively oriented surface is one in which the normal vectors all point outward or away from the surface. For standardization, flux (or *flux-out*) implies through a positively oriented surface.

EXAMPLE 1: Find the flux of $\mathbf{F} = \langle y, x, z \rangle$ across the boundary of the solid region E enclosed by the paraboloid $z = 3 - x^2 - y^2$ and the plane $z = 2$.

EXAMPLE 2: Find the flux of $\mathbf{F} = \langle z, y, x \rangle$ across the unit sphere $x^2 + y^2 + z^2 = 1$.

INSERT FLUX ARTICLE HERE

Q203: Chapter 16B: Lesson 3 – The Divergence Theorem

Let E be a region in three dimensions bounded by a closed surface S, and let $\hat{n}$ denote the unit outer normal vector to S at (x, y, z). If **F** is a vector function that has continuous partial derivatives on E, then :

$$\iint_S \mathbf{F} \cdot \hat{n} dS = \iiint_E \nabla \cdot \mathbf{F} \, dV$$

In other words, the flux of **F** over S equals the triple integral of the divergence of **F** over E.

1. Let E be the region bounded by the cylinder $x^2 + y^2 = 4$ and the planes $z = 0$ and $z = 3$, and let S denote the surface of E. If $\mathbf{F} = \langle x^3, y^3, z^3 \rangle$, use the divergence theorem to find $\iint_S \mathbf{F} \cdot \hat{n} dS$.

2. Let E be the region bounded by the cylinder $z = 4 - x^2$, the plane $y + z = 5$ and the xy – and xz – planes, and let S be the surface of E. If $\mathbf{F} = \left\langle x^3 + \sin z, x^2 y + \cos z, e^{x^2+y^2} \right\rangle$, use the divergence theorem to find $\iint_S \mathbf{F} \cdot \hat{n} dS$.

3. **CHALLENGE**

An inverse square field is given by $\mathbf{F} = \left(q / r^3\right)\mathbf{r}$, where q is a constant, $\mathbf{r} = \langle x, y, z \rangle$, and $|\mathbf{r}| = r$.
It can be shown that the divergence of every inverse square field is zero.
Now suppose a closed surface S forms the boundary of a region E which encloses the origin.
Prove the flux of **F** over S is $4\pi q$ regardless of the shape of E.

Q203: Chapter 16B: Lesson 4 – Stokes' Theorem

Introduction:

Additional Analysis

1. Let S be the part of the paraboloid $z = 9 - x^2 - y^2$ with $z \geq 0$, and let C be the trace of S on the xy-plane. Verify Stokes' Theorem for the vector field $\mathbf{F} = \langle 3z, 4x, 2y \rangle$.

2. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = \langle -y^2, x, z^2 \rangle$ and C is the curve of intersection of the plane $y + z = 2$ and the cylinder $x^2 + y^2 = 1$. (Orient C to be counterclockwise when viewed from above.)

3. Use Stokes' Theorem to compute $\iint_S curl\mathbf{F} \cdot \hat{n} dS$, where $\mathbf{F} = \langle xz, yz, xy \rangle$ and S is the part of the sphere $x^2 + y^2 + z^2 = 4$ that lies inside the cylinder $x^2 + y^2 = 1$ and above the *xy*-plane.

C3: Q203 CH16B LESSON 5: DISCUSSION, INSIGHT AND REVIEW

C3: Q203 CH16B LESSON 6: REVIEW PRACTICE PROBLEMS