BC.Q204.L4. REVIEW SOLUTIONS

#1 $\frac{dy}{dx} = 6(14-y)$

A. $\int \frac{dy}{14-y} = \int 6\,dx$

$-\ln|14-y| = 6x + C$

$\ln|14-y| = -6x + C$

$|14-y| = Ce^{-6x}$

$14 - y = C^* e^{-6x}$

$y = 14 - C^* e^{-6x}$

$2 = 14 - C^* \cancel{e^{0}}^{\,1}$

$\therefore C^* = 12$

$\boxed{y = 14 - 12e^{-6x}}$

B. $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(6(14-y)\right) = \frac{d}{dx}\left(84-6y\right) = \overset{\text{implicit}}{-6\frac{dy}{dx}} \overset{\text{sub}}{=} -6\left(84-6y\right)$

$\left.\frac{d^2y}{dx^2}\right|_{y=4} = -6(84-6(4)) = -360$ ✓

OR

$\left.\frac{d^2y}{dx^2}\right|_{y=4,\ \frac{dy}{dx}=} -6(60) = -360$ ✓

#2 $\frac{dy}{dx} = 6(14+y)$

A. $\int \frac{dy}{14+y} = \int 6\,dx$

$\ln|14+y| = 6x + C$

$|14+y| = Ce^{6x}$

$14 + y = C^* e^{6x}$

$y = -14 + C^* e^{6x}$

$2 = -14 + C^* \cancel{e^{0}}^{\,1}$

$C^* = 16$

$\boxed{y = -14 + 16e^{6x}}$

B. $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(84+6y\right) = 6\frac{dy}{dx} = 6\left(84+6y\right)\Big|_{y=4} = 648$ ✓

#3 $\frac{dy}{dx} = 3x^2y^2$

$\int \frac{dy}{y^2} = \int 3x^2\,dx$

$\int y^{-2}\,dy = \int 3x^2\,dx$

$\frac{y^{-1}}{-1} = x^3 + c$

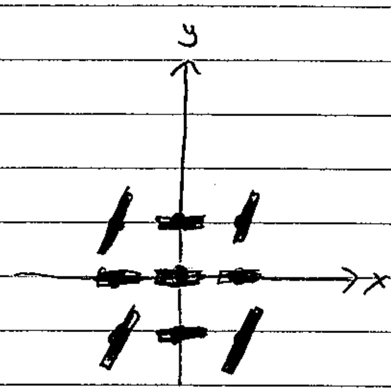
$\frac{-1}{y} = x^3 + c$

$\frac{1}{y} = -x^3 + c$

$y = \frac{1}{-x^3+c}$

$\frac{1}{3} = \frac{1}{0+c} \quad \therefore c = 3$

$\boxed{y = \frac{1}{-x^3+3}}$



D: $-x^3 + 3 \neq 0 \quad x^3 \neq 3$

$-x^3 \neq -3 \quad \{x \neq \sqrt[3]{3}\}$

4

$\frac{dy}{dx} = x^4 - 3y^2 + 6$ CANNOT SEPARATE!

$f(0) = 1$

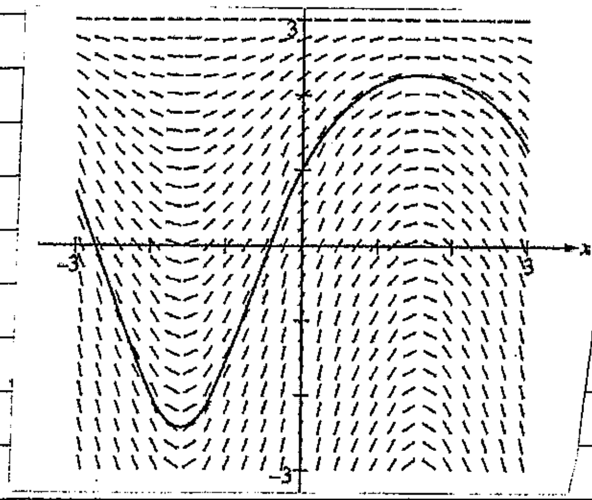
$f'(0,1) = -3(1)^2 + 6 = \boxed{3}$

$f''(x,y) = 4x^3 - 6y\frac{dy}{dx}$

$\left.\frac{d^2y}{dx^2}\right|_{x=0\ y=1,\ \frac{dy}{dx}=3} = 4(0)^3 - 6(1)(3) = \boxed{-18}$

$f''(0)$

5



(b) $\left.\frac{dy}{dx}\right|_{x=0,\ y=1} = (3-1)\cos(0) = \boxed{2}$

$L(x) = f(0) + f'(0,1)(x-0)$

$L(x) = 1 + 2(x)$

$f(0.2) \approx L(0.2) = 1 + 2(0.2) = 1 + 0.4 = \boxed{1.4}$

(c) $\int \frac{dy}{3-y} = \int \cos x\, dx$

$-\ln|3-y| = \sin x + c$

$\ln|3-y| = -\sin x + c$

$|3-y| = ce^{-\sin x}$

$3-y = c^* e^{-\sin x}$

$y = 3 - c^* e^{-\sin x}$

$1 = 3 - c^* \cancel{e^{0}}^{\,1}$ $\quad c^* = 2$

$\boxed{y = 3 - 2e^{-\sin x}}$

6. Consider the differential equation $\frac{dy}{dx} = x^2 - \frac{1}{2}y$

Let $y = h(x)$ be the particular solution to the differential equation with $h(0) = 2$.

A. Use a linearization centered at $x = 0$ to approximate $h(1)$.

B. Use Euler's method, starting at $x = 0$ with two steps of equal size, to approximate $h(1)$.

A] $L(x) = h(0) + h'(0,2)(x-0)$

$L(x) = 2 + \left[0^2 - \frac{1}{2}(2)\right](x)$

$L(x) = 2 - x$

$h(1) \approx L(1) = 2 - 1 = \boxed{1}$

C] $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(x^2 - \frac{y}{2}\right) = 2x - \frac{1}{2}\frac{dy}{dx}$

$\left.\frac{d^2y}{dx^2}\right|_{x=0,\ y=2,\ \frac{dy}{dx}=-1} = 2(0) - \frac{1}{2}(-1) = \frac{1}{2} > 0$

∴ The graph of f at the point (0,2) is concave up.

$\Delta x = 0.5$

B] $(\overset{x_0}{0}, \overset{y_0}{2})$ $\quad \hat{y}_1 = 2 + \left[0^2 - \frac{1}{2}(2)\right]\overset{\Delta x}{0.5}$

$= 2 - [1]0.5$

$= \frac{3}{2}$

$(0.5, \frac{3}{2})$ $\quad \hat{y}_2 = \frac{3}{2} + \left[\left(\frac{1}{2}\right)^2 - \frac{1}{2}\left(\frac{3}{2}\right)\right]\overset{\Delta x}{0.5}$

$= \frac{3}{2} + \left[\frac{1}{4} - \frac{3}{4}\right]\frac{1}{2}$

$= \frac{3}{2} + \left[-\frac{1}{2}\right]\frac{1}{2}$

$= \frac{3}{2} - \frac{1}{4}$

$= \frac{6}{4} - \frac{1}{4}$

$= \boxed{\frac{5}{4}}$

$\boxed{h(1) \approx E(1) = \frac{5}{4}}$

7. Consider the differential equation $\frac{dy}{dx} = y^2(2x+2)$

Let $y = f(x)$ be the particular solution to the differential equation with $f(0) = -1$.

A. Use Euler's method, starting at $x = 0$ with two steps of equal size, to approximate $f\left(\frac{1}{2}\right)$.

$\Delta x = \frac{1}{4}$

B. Find $y = f(x)$, the particular solution to the differential equation with initial condition $f(0) = -1$.

A]. $x_0 \ y_0$

$(0, -1)$

$$\hat{y}_1 = -1 + \left[(-1)^2(2(0)+2)\right]\frac{1}{4}$$
$$= -1 + [2]\frac{1}{4}$$
$$= -1 + \frac{1}{2} = -\frac{1}{2}$$

$\left(\frac{1}{4}, -\frac{1}{2}\right)$

$$\hat{y}_2 = -\frac{1}{2} + \left[\left(\frac{-1}{2}\right)^2\left(2\frac{1}{4}+2\right)\right]\frac{1}{4}$$
$$= -\frac{1}{2} + \left[\frac{1}{4}\left(\frac{5}{2}\right)\right]\frac{1}{4}$$
$$= -\frac{1}{2} + \left[\frac{5}{32}\right] = -\frac{16}{32} + \frac{5}{32} = \frac{-11}{32}$$

$$f\left(\tfrac{1}{2}\right) \approx E\left(\tfrac{1}{2}\right) = \boxed{\frac{-11}{32}}$$

B]

$$\int \frac{dy}{y^2} = \int (2x+2)\,dx$$

$$\int y^{-2}\,dy = \int (2x+2)\,dx$$

$$\frac{y^{-1}}{-1} = x^2 + 2x + c$$

$$\frac{-1}{y} = -x^2 - 2x + c$$

$$y = \frac{1}{-x^2 - 2x + c}$$

$$-1 = \frac{1}{c} \quad \therefore c = -1$$

$$y = \frac{1}{-x^2-2x-1} \quad \text{or} \quad \frac{-1}{x^2+2x+1}$$

#8 : If $\frac{dP}{dt} = 0.001P(25-P)$ and $P(0) = 15\ldots$ $\quad K = 0.001 \quad L = 25$

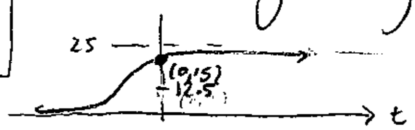
A. Find the function $P(t)$. Provide a quick sketch.

B. Find the value of P when $\frac{dP}{dt}$ is at its greatest.

$$P = \frac{25}{1+c^* e^{-0.025t}} \rightarrow 15 = \frac{25}{1+c^*} \rightarrow 1+c^* = \frac{25}{15} \rightarrow c^* = \frac{5}{3} - 1 = \frac{2}{3}$$

A. $\boxed{P = \dfrac{25}{1+\frac{2}{3}e^{-0.025t}}}$

B. P is growing fastest $P = \frac{25}{2} = \boxed{12.5}$



#9 : If $\frac{dP}{dt} = 40P - 5P^2$ and $P(0) = 3\ldots$

A. Find the function $P(t)$. $\longrightarrow \frac{dP}{dt} = 5P(8-P) \quad K=5 \quad L=8$

B. Find the value of $\frac{dP}{dt}$ when $\frac{dP}{dt}$ is at its greatest.

$$P = \frac{8}{1+c^* e^{-40t}} \rightarrow 3 = \frac{8}{1+c^*} \rightarrow 1+c^* = \frac{8}{3} \rightarrow c^* = \frac{5}{3}$$

A. $\boxed{P = \dfrac{8}{1+\frac{5}{3}e^{-40t}}}$

B. P is growing fastest when $P = 4$ and $\frac{dP}{dt} = 40(4) - 5(16) = \boxed{80}$

#10 : If $\frac{40}{Z}\frac{dZ}{dt} = 5 - \frac{Z}{60}$ and $Z(0) = 25\ \ldots \rightarrow \frac{dZ}{dt} = \frac{Z}{40}\left(5 - \frac{Z}{60}\right) \rightarrow \frac{dZ}{dt} = \frac{Z}{2400}(300 - Z)$

A. Find the function $Z(t)$.

B. Find $\lim_{t\to\infty} Z(t)$. $\quad K = \frac{1}{2400} \quad L = 300$

$$Z = \frac{300}{1+c^* e^{-\frac{1}{8}t}} \rightarrow 25 = \frac{300}{1+c^*} \rightarrow 1+c^* = 12 \rightarrow c^* = 11$$

$\boxed{Z = \dfrac{300}{1+11e^{-\frac{1}{8}t}}}$

B. $\lim_{t\to\infty} Z(t) = \underline{300}$

#) 11

$$\frac{dR}{dt} = kR \qquad 24 = C^* \cancel{e^0}^{\,1} \quad \therefore\ C^* = 24$$

$$\int \frac{dR}{R} = \int k\,dt \qquad (a)\ \boxed{R = 24e^{kt}}$$

$$\ln|R| = kt + C \qquad (b)\ 72 = 24e^{2k}$$

$$|R| = Ce^{kt}$$

$$R = C^* e^{kt} \qquad 3 = e^{2k} \longrightarrow \ln(3) = 2k \longrightarrow \boxed{k = \frac{\ln(3)}{2}}$$

$$(c)\quad R = 24e^{\frac{\ln(3)}{2}t} \Rightarrow R = 24e^{\ln(3)\cdot\frac{t}{2}}$$

$$\boxed{R = 24(3)^{\frac{t}{2}}} \qquad R(8) = 24(3)^4 = \boxed{1944 \text{ RATS}}$$

12.

$$\frac{dT}{dt} = k(20 - T)$$

A.
$$\int \frac{dT}{20 - T} = \int k\,dt$$

$$-\ln|20 - T| = kt + C$$

$$\ln|20 - T| = -kt + C \longrightarrow |20 - T| = Ce^{-kt}$$

$$20 - T = C^* e^{-kt}$$

$$T = 20 - C^* e^{-kt}$$

$$100 = 20 - C^* \cancel{e^0}^{\,1} \qquad C^* = -80$$

$$\boxed{T = 20 + 80e^{-kt}}$$

B.
$$60 = 20 + 80e^{-3k}$$

$$40 = 80e^{-3k}$$

$$\frac{1}{2} = e^{-3k}$$

$$\ln(\tfrac{1}{2}) = -3k$$

$$\boxed{k = \frac{\ln(1/2)}{-3}}$$

C.
$$T = 20 + 80e^{-\frac{\ln(1/2)}{-3}t}$$

$$= 20 + 80e^{\ln(1/2)\cdot\frac{t}{3}}$$

$$\boxed{T = 20 + 80\left(\tfrac{1}{2}\right)^{t/3}}$$

$$T(6) = 20 + 80\left(\tfrac{1}{2}\right)^2$$

$$= 20 + 80\left(\tfrac{1}{4}\right) = 20 + 20$$

$$= \boxed{40°C}$$

D.
$$30 = 20 + 80\left(\tfrac{1}{2}\right)^{t/3} \longrightarrow \ln\left(\tfrac{1}{8}\right) = \tfrac{t}{3}\ln\left(\tfrac{1}{2}\right)$$

$$10 = 80\left(\tfrac{1}{2}\right)^{t/3} \qquad \frac{t}{3} = \frac{\ln(1/8)}{\ln(1/2)} \longrightarrow t = \frac{3\ln(1/8)}{\ln(1/2)} \text{ min}$$

$$\frac{1}{8} = \left(\frac{1}{2}\right)^{t/3}$$

$$t = \frac{3\ln(1/8)}{\ln(1/2)} = \frac{3(-\ln(8))}{-\ln(2)} = \frac{3\ln(8)}{\ln(2)} = \frac{3\ln(2)^3}{\ln(2)} = \frac{9\ln(2)}{\ln(2)} = \boxed{9 \text{ min}}$$