AB Q304 LESSON 1B

FUNDAMENTAL THEOREM OF CALCULUS

PART 1

I. THE FUNDAMENTAL THEOREM OF CALCULUS PART 1 (Defines the Relationship)

Defining the relationship: ~~[scribbled out]~~ $\int f(x)\,dx = g(x) + C$

$\left[\begin{array}{l} g \text{ is an antiderivative of } f(x) \\ f \text{ is the derivative of } g(x) \end{array}\right\} \quad g'(x) = f(x)$

ex: $g(x) = \int x^2\,dx \qquad g'(x) = x^2$

ex: $g(x) = \int \sin(e^x)\,dx \qquad g'(x) = \sin(e^x)$

Defining the relationship: $g(x) = \int_a^x f(t)\,dt$ (a is a constant)

$\left[\begin{array}{l} g \text{ is a particular antiderivative of } f(x) \\ \qquad\qquad \text{with } g(a) = 0 \\ \\ f \text{ is the derivative of } g(x) \end{array}\right\} \quad g'(x) = f(x)$

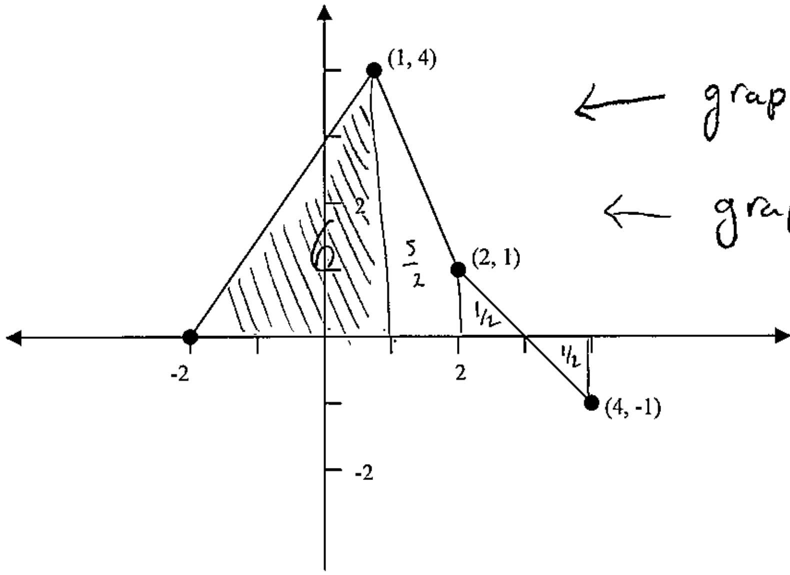
ex: $g(x) = \int_1^x t^2\,dt \qquad g'(x) = x^2$

verify: $g(x) = \int_1^x t^2\,dt = \left[\frac{t^3}{3}\right]_1^x = \frac{x^3}{3} - \frac{1}{3} \qquad g'(x) = x^2$ ✓

ex: $g(x) = \int_1^x \sin(e^t)\,dt \qquad g'(x) = \sin(e^t)$

Go with Theory not able to verify.

GUIDED PRACTICE 1: 1999 #5 (adaptation)



← graph of f(x)

← graph of g'(x)

Derivative Graph

$* \; g'(x) = f(x)$

EX1: The graph of the function *f*, consisting of three line segments, is given above. Let $g(x) = \int_1^x f(t)dt$.

A. Compute g(4) and g(-2)
B. Find the instantaneous rate of change of g, with respect to x, at x = 1.
C. Find the absolute maximum and minimum values of g on the closed interval [-2, 4]. Justify your answer.
D. The second derivative of g is not defined at x = 1 and x = 2. How many of these values are x-coordinates of points of inflection of the graph of g? Justify your answer.

A] $g(4) = \int_1^4 f(x)dx = \frac{5}{2} + \frac{1}{2} - \frac{1}{2} = \frac{5}{2}$

$g(-2) = \int_1^{-2} f(x)dx = -\int_{-2}^{1} f(x)dx = -[6] = -6$

B]. $g'(1) = f(1) = 4$ ← point on derivative graph

C] interior critical $g'(x) = 0$ at $x = 3$
end point critical at $x = -2$; $x = 4$

$g(-2) = -6$

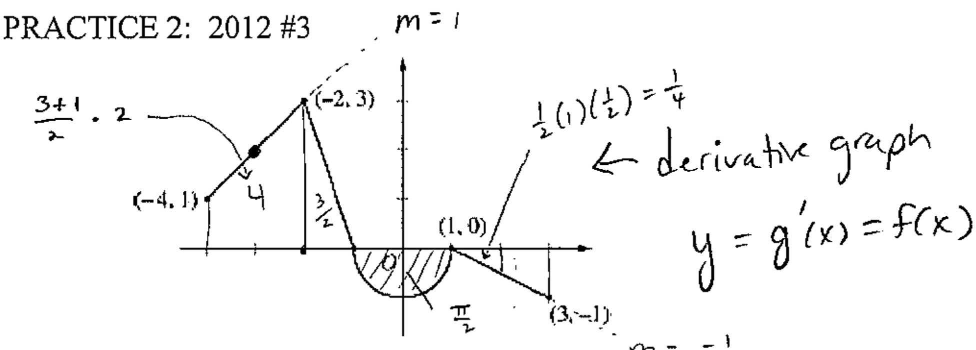
$g(3) = \int_1^3 f(x)dx = \frac{5}{2} + \frac{1}{2} = 3$

$g(4) = \frac{5}{2}$

$\boxed{\text{ABS MAX} = 3 \quad \text{ABS MIN} = -6}$

D] g has a point of inflection at x = 1 b/c $g''(x) = f'(x)$ changes sign at this x-value.

GUIDED PRACTICE 2: 2012 #3



Graph of f

Let f be the continuous function defined on $[-4, 3]$ whose graph, consisting of three line segments and a semicircle centered at the origin, is given above. Let g be the function given by $g(x)=\int_1^x f(t)\,dt$.

(a) Find the values of $g(2)$ and $g(-2)$.

(b) For each of $g'(-3)$ and $g''(-3)$, find the value or state that it does not exist.

(c) Find the x-coordinate of each point at which the graph of g has a horizontal tangent line. For each of these points, determine whether g has a relative minimum, relative maximum, or neither a minimum nor a maximum at the point. Justify your answers.

(d) For $-4 < x < 3$, find all values of x for which the graph of g has a point of inflection. Explain your reasoning.

(a) $g(2)=\int_1^2 f(x)\,dx = -\frac{1}{4}$

$g(-2)=\int_1^{-2} f(x)\,dx = -\int_{-2}^{1} f(x)\,dx = -\left[\frac{3}{2}-\frac{\pi}{2}\right]$

b) $g'(-3)=f(-3)=2$ ← point on derivative graph

$g''(-3)=f'(-3)=1$ ← slope of derivative graph

c) g has a horizontal tangent when $g'(x)=0$ at $x=-1, x=1$

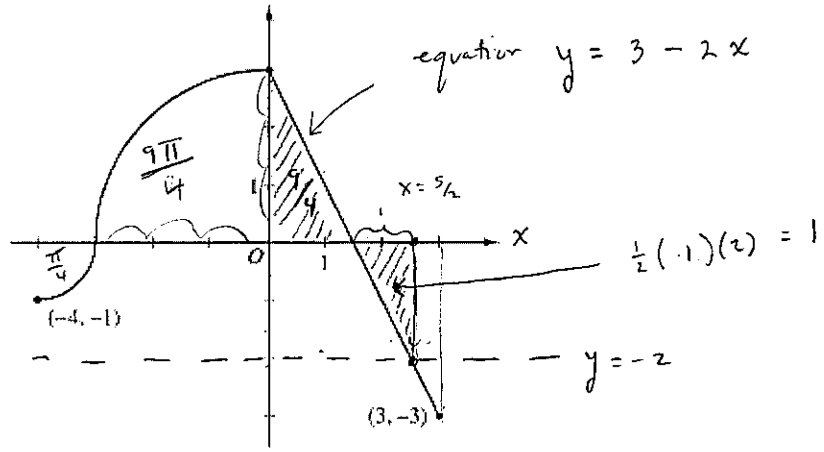
AT $x=-1$: g has a relative max b/c $g'(x)$ goes from positive to negative

AT $x=1$: g has neither a max or min b/c $g'(x)$ does not change signs

D) g has a point of inflection at $x=-2, x=0, x=1$ b/c at these x-values $g''(x)=f'(x)$ changes sign.

$\frac{1}{2}(3)(\frac{3}{2}) = \frac{9}{4}$

GUIDED PRACTICE 3: 2011 #4



Graph of f

The continuous function f is defined on the interval $-4 \le x \le 3$. The graph of f consists of two quarter circles and one line segment, as shown in the figure above. Let $g(x) = 2x + \int_0^x f(t)\,dt$.

(a) Find $g(-3)$. Find $g'(x)$ and evaluate $g'(-3)$.

(b) Determine the x-coordinate of the point at which g has an absolute maximum on the interval $-4 \le x \le 3$. Justify your answer.

(c) Find all values of x on the interval $-4 < x < 3$ for which the graph of g has a point of inflection. Give a reason for your answer.

(d) Find the average rate of change of f on the interval $-4 \le x \le 3$. There is no point c, $-4 < c < 3$, for which $f'(c)$ is equal to that average rate of change. Explain why this statement does not contradict the Mean Value Theorem.

(a) $g(-3) = 2(-3) + \int_0^{-3} f(x)\,dx = -6 - \int_{-3}^{0} f(x)\,dx = -6 - \frac{9\pi}{4}$

$g'(x) = 2 + f(x)$; $g'(-3) = 2 + f(-3) = 2 + 0 = 2$

(b) interior critical : $g'(x) = 2 + f(x) = 0 \rightarrow f(x) = -2$ $\quad 3 - 2x = -2$

endpoint critical $x = -4$, $x = 3$ $\quad 2x = 5$

$x = 5/2$

$g(-4) = 2(-4) - \int_{-4}^{0} f(x)\,dx = -8 + \frac{\pi}{4} \cancel{+} - \frac{9\pi}{4}$

$g(5/2) = 2(\frac{5}{2}) + \int_0^{5/2} f(x)\,dx = 5 + \frac{9}{4} - 1 = 5 + \frac{9}{4} - \frac{4}{4} = \boxed{5 + \frac{5}{4}} = 6.25$

$g(3) = 2(3) + \int_0^3 f(x)\,dx = 6 + \frac{9}{4} - \frac{9}{4} = 6$

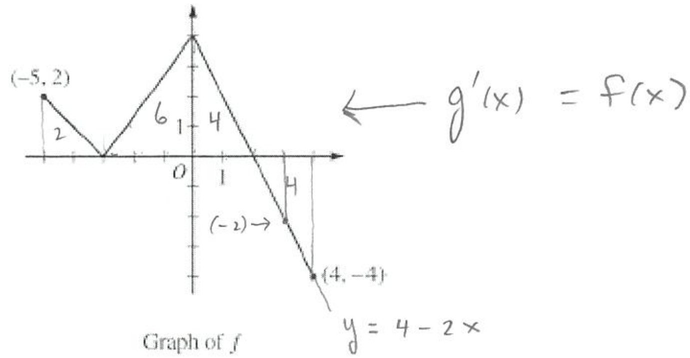
(c) g has a point of inflection at $x = 0$ b/c $g''(x) = f'(x)$ changes sign at this x-value.

(d) Ave rate Δ of f $= \frac{f(3) - f(-4)}{3 - (-4)} = \frac{-3 - (-1)}{7} = \frac{-2}{7}$

f is not differentiable on $(-4, 3)$ as there is a corner at $x = 0$

M.V.T DOES NOT GUARANTEE.

GUIDED PRACTICE 4: 2014 #3



Graph of f

The function f is defined on the closed interval $[-5, 4]$. The graph of f consists of three line segments and is shown in the figure above. Let g be the function defined by $g(x) = \int_{-3}^{x} f(t)\,dt$.

(a) Find $g(3)$.

(b) On what open intervals contained in $-5 < x < 4$ is the graph of g both increasing and concave down? Give a reason for your answer.

(c) The function h is defined by $h(x) = \frac{g(x)}{5x}$. Find $h'(3)$.

(d) The function p is defined by $p(x) = f(x^2 - x)$. Find the slope of the line tangent to the graph of p at the point where $x = -1$.

(a) $g(3) = \int_{-3}^{3} f(x)\,dx = 6 + 4 - \frac{1}{2}(1)(2) = 6 + 4 - 1 = \boxed{9}$ $\boxed{+1}$

(b) $x \in (-5, -3) \cup (0, 2)$ b/c on this interval both $\boxed{+1}$

$g'(x) = f(x)$ is positive (above the x-axis) $\boxed{+1}$

and $g''(x) = f'(x)$ is negative (slope of f is negative)

$g'(3) = -2$

(c) $h'(x) = \frac{5x\, g'(x) - g(x) \cdot 5}{(5x)^2}$ $\boxed{+2}$ $h'(3) = \frac{15\, g'(3) - g(3) \cdot 5}{(15)^2}$

$h'(3) = \boxed{\frac{15(-2) - 9(5)}{(15)^2}} = -\frac{1}{3}$ $\boxed{+1}$

(d) $p'(x) = f'(x^2 - x)(2x - 1)$ $\boxed{+2}$

$p'(-1) = f'(1 + 1)(-2 - 1)$

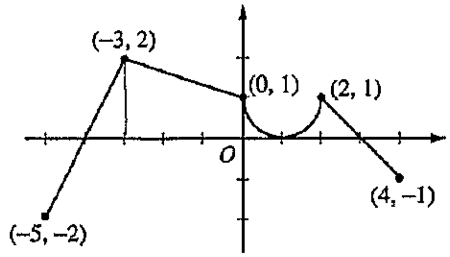
$= f'(2)(-3)$

$= (-2)(-3)$

$= \boxed{6}$ $\boxed{+1}$

OTHER PRACTICE 1

HW1: 2004 #5



Graph of f

$g'(x) = f(x)$

5. The graph of the function f shown above consists of a semicircle and three line segments. Let g be the function given by $g(x) = \int_{-3}^{x} f(t)\,dt$.

   (a) Find $g(0)$ and $g'(0)$.

   (b) Find all values of x in the open interval $(-5, 4)$ at which g attains a relative maximum. Justify your answer.

   (c) Find the absolute minimum value of g on the closed interval $[-5, 4]$. Justify your answer.

   (d) Find all values of x in the open interval $(-5, 4)$ at which the graph of g has a point of inflection.

(a) $g(0) = \int_{-3}^{0} f(x)\,dx = \frac{2+1}{2} \cdot 3 = \frac{9}{2}$ $\quad g'(0) = f(0) = 1 \leftarrow$ point on graph

(b) g has a relative max at $x = 3$ b/c $g'(x) = f(x) = 0$ at $x = 3$ and $g'(x)$ goes from positive to negative at $x = 3$.

(c) interior critical x : $g'(x) = 0$ at $x = -4, 1, 3$

$g(1) \rightarrow$ NOT A CANDIDATE $g'(x)$ does not change sign at $x = 1$

$g(3) \rightarrow$ NOT A CANDIDATE g has a rel. max at $x = 3$

endpoint critical x : $x = -5, 4$

$g(-5) = \int_{-3}^{-5} f(x)\,dx = -\int_{-5}^{-3} f(x)\,dx = -(0) = 0$

$g(-4) = \int_{-3}^{-4} f(x)\,dx = -\int_{-4}^{-3} f(x)\,dx = -\left(\frac{1}{2} \cdot 2\right) = -1$

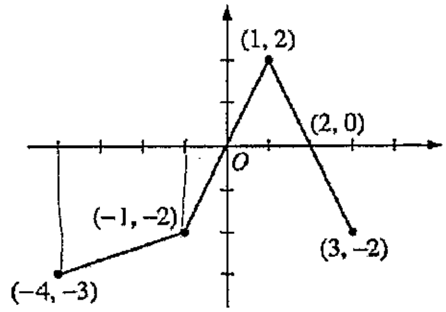
$g(4) = \int_{-3}^{4} f(x)\,dx = \frac{9}{2} + \int_{0}^{4} f(x)\,dx = \frac{9}{2} + \left(2 - \frac{\pi}{2}\right) + \frac{1}{2} - \frac{1}{2} > 0$

$\boxed{-1}$ is the absolute minimum of $g(x)$ on $[-5, 4]$

(D) g has a point of inflection at $x = -3, 1, 2$ b/c $g''(x) = f'(x)$ changes sign at these x-values

OTHER PRACTICE 2

HW 2: 2005FB #4



Graph of f

4. The graph of the function f above consists of three line segments.

(a) Let g be the function given by $g(x) = \int_{-4}^{x} f(t)\,dt$. For each of $g(-1)$, $g'(-1)$, and $g''(-1)$, find the value or state that it does not exist.

(b) For the function g defined in part (a), find the x-coordinate of each point of inflection of the graph of g on the open interval $-4 < x < 3$. Explain your reasoning.

(c) Let h be the function given by $h(x) = \int_{x}^{3} f(t)\,dt$. Find all values of x in the closed interval $-4 \le x \le 3$ for which $h(x) = 0$.

(d) For the function h defined in part (c), find all intervals on which h is decreasing. Explain your reasoning.

(a) $g(-1) = \int_{-4}^{-1} f(x)dx = -\left[\frac{3+2}{2} \cdot 3\right] = \frac{-15}{2}$ appeal to geometry

$g'(-1) = f(-1) = -2$ point on graph

$g''(-1) = f'(-1) \to DNE$ corner on $f(x)$ graph

(b) g has a point of inflection at $x = 1$ only b/c $g''(x) = f'(x)$ changes sign at this x value.

(c) $h(x) = \int_{x}^{3} f(t)dx = -\int_{3}^{x} f(x)dt = 0$ when $\int_{3}^{x} f(t)dt = 0$ or $\int_{x}^{3} f(t)dt = 0$

at $x = -1$: $\int_{-1}^{3} f(x)dx = 0$ appeal to geometry

at $x = 1$: $\int_{1}^{3} f(x)dx = 0$ appeal to geometry

at $x = 3$: $\int_{3}^{3} f(x)dx = 0$ property

(d) $h(x) = -\int_{3}^{x} f(t)dt$ $h'(x) = -f(x)$

$h'(x) = 0$ when $-f(x) = 0$ which occurs at $x = 0, 2$

$h(x)$ is decreasing when $h'(x) = -f(x) < 0$. This occurs on the interval $[0, 2]$.

$f(x)$ is above the x-axis so $-f(x)$ is negative.