AB : Q302 HW SOLUTIONS — LESSON 1

1. $\frac{dy}{dt} = 5(10-y)$

$\int \frac{dy}{10-y} = \int 5\,dt$

$-\ln|10-y| = 5t + c$

$\ln|10-y| = -5t + c$

$|10-y| = Ce^{-5t}$

$10-y = C^* e^{-5t}$

$y = 10 - C^* e^{-5t}$

$7 = 10 - C^* e^0 \quad \therefore C^* = 3$

$\boxed{y = 10 - 3e^{-5t}}$

2. $\frac{dy}{dt} = \frac{1}{2}(5-y)$

$\int \frac{dy}{5-y} = \int \frac{1}{2}\,dt$

$-\ln|5-y| = \frac{1}{2}t + c$

$\ln|5-y| = \frac{-t}{2} + c$

$|5-y| = Ce^{-t/2}$

$5-y = C^* e^{-t/2}$

$y = 5 - C^* e^{-t/2}$

$8 = 5 - C^* e^0 \quad \therefore C^* = -3$

$\boxed{y = 5 + 3e^{-t/2}}$

3. $\frac{dy}{dt} = 8(2+y)$

$\int \frac{dy}{2+y} = \int 8\,dt$

$\ln|2+y| = 8t + C$

$|2+y| = Ce^{8t}$

$2+y = C^* e^{8t}$

$y = -2 + C^* e^{8t}$

$1 = -2 + C^* e^0 \quad C^* = 3$

$\boxed{y = -2 + 3e^{8t}}$

4. $\frac{dy}{dt} = -4(3-y)$

$\int \frac{dy}{3-y} = \int -4\,dt$

$-\ln|3-y| = -4t + C$

$\ln|3-y| = 4t + c$

$|3-y| = Ce^{4t}$

$3-y = C^* e^{4t}$

$y = 3 - C^* e^{4t}$

$6 = 3 - C^* e^0 \quad C^* = -3$

$\boxed{y = 3 + 3e^{4t}}$

5. $\frac{dy}{dt} = 0.4y$

$\int \frac{dy}{y} = \int 0.4\,dt$

$\ln|y| = 0.4t + c$

$|y| = Ce^{0.4t}$

$y = C^* e^{0.4t}$

$6 = C^* e^0 \quad \therefore C^* = 6$

$\boxed{y = 6e^{0.4t}}$

① $\frac{dy}{dx} = \frac{x}{y}$

$\int y\,dy = \int x\,dx$

$\frac{y^2}{2} = \frac{x^2}{2} + C$

$y^2 = x^2 + C$

$y = \pm\sqrt{x^2 + C}$

$2 = +\sqrt{1 + c}$ $\quad \therefore\ C = 3$

$\boxed{y = \sqrt{x^2 + 3}}$ $\quad x \in \mathbb{R}$

② $\frac{dy}{dx} = \frac{y}{x}$

$\int \frac{dy}{y} = \int \frac{dx}{x}$

$\ln|y| = \ln|x| + C$

$|y| = C e^{\ln|x|}$

$|y| = C|x|$

$y = C^*|x|$

$2 = C^*|2|$ $\quad \therefore\ C^* = 1$

$\boxed{y = |x|}$ $\quad x \in R$

③ $\frac{dy}{dx} = xy$

$\int \frac{dy}{y} = \int x\,dx$

$\ln|y| = \frac{x^2}{2} + c$

$|y| = Ce^{x^2/2}$

$y = C^* e^{x^2/2}$

$2 = C^* e^{1/2}$ $\quad \therefore\ C^* = \frac{2}{e^{1/2}}$ or $2e^{-1/2}$

$\boxed{y = 2e^{-1/2} \cdot e^{x^2/2}}$

or

$\boxed{y = 2e^{\frac{x^2-1}{2}}}$ $\quad x \in \mathbb{R}$

④ $\frac{dy}{dx} = \frac{1}{xy}$

$\int y\,dy = \int \frac{1}{x}\,dx$

$\frac{y^2}{2} = \ln|x| + c$

$y^2 = 2\ln|x| + c$

$y = \pm\sqrt{2\ln|x| + c}$

$-4 = -\sqrt{2\ln(1) + c}$

$4 = \sqrt{c}$ $\quad \therefore\ c = 16$

$\boxed{y = -\sqrt{2\ln|x| + 16}}$ $\quad x \in \mathbb{R}$

⑤ $\frac{dy}{dx} = (y+5)(x+2)$

$\int \frac{dy}{y+5} = \int (x+2)\,dx$

$\ln|y+5| = \frac{x^2}{2} + 2x + C$

$|y+5| = Ce^{\left(\frac{x^2}{2}+2x\right)}$

$y+5 = C^* e^{\left(\frac{x^2}{2}+2x\right)}$

$y = -5 + C^* e^{\left(\frac{x^2}{2}+2x\right)}$

$1 = -5 + C^* e^0$

$\therefore\ C^* = 6$

$\boxed{y = -5 + 6e^{\left(\frac{x^2}{2}+2x\right)}}$

$x \in \mathbb{R}$

(6) $\frac{dy}{dx} = (\cos x)e^{(y+\sin x)} = (\cos x)e^{y} \cdot e^{(\sin x)}$

$\boxed{u = \sin x}$

$\int \frac{dy}{e^y} = \int (\cos x) e^{(\sin x)} dx \longrightarrow \int e^{-y} dy = \int e^u du$

$-e^{-y} = e^u + C \longrightarrow -e^{-y} = e^{\sin x} + C$

$e^{-y} = -e^{\sin x} + C \longrightarrow -y = \ln(-e^{\sin x} + c)$

Key Fact $\boxed{\ln(1) = 0}$

$y = -\ln(-e^{\sin x} + c) \rightarrow 0 = -\ln(-e^0 + c)$

$0 = -\ln(-1 + C) \quad \therefore C = 2$

$y = -\ln(-e^{\sin x} + 2)$ or $y = -\ln(2 - e^{\sin x})$

Domain: $\{2 - e^{\sin x} > 0\} \equiv \{e^{\sin x} < 2\} \equiv \{\sin x < \ln(2)\}$

(7) $\frac{dy}{dx} = -2xy^2$

$\int \frac{dy}{y^2} = \int -2x\, dx$

$\int y^{-2} dy = \int -2x\, dx$

$\rightarrow -y^{-1} = -x^2 + c$

$y^{-1} = x^2 + c$

$\frac{1}{y} = x^2 + C$

$\rightarrow y = \frac{1}{x^2 + C}$

$\frac{1}{4} = \frac{1}{1 + c} \quad \therefore c = 3$

$\boxed{y = \frac{1}{x^2 + 3}} \quad x \in \mathbb{R}$